On negative flows of the AKNS hierarchy and a class of deformations of a bihamiltonian structure of hydrodynamic type

2006 J. Phys. A: Math. Gen. 39 1099

(http://iopscience.iop.org/0305-4470/39/5/006)

Institute of Physics Publishing — Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **39** (2006) 1099–1114 doi:10.1088/0305-4470/39/5/006

On negative flows of the AKNS hierarchy and a class of deformations of a bihamiltonian structure of hydrodynamic type

H Aratyn[1], J F Gomes[2] and A H Zimerman[2]

[1] Department of Physics, University of Illinois at Chicago, 845 W Taylor Street, Chicago, IL 60607-7059, USA

[2] Instituto de Física Teórica-UNESP, Rua Pamplona 145, 01405-900 São Paulo, Brazil

Received 7 October 2005, in final form 9 October 2005
Published 18 January 2006
Online at stacks.iop.org/JPhysA/39/1099

Abstract

A deformation parameter of a bihamiltonian structure of hydrodynamic type is shown to parametrize different extensions of the AKNS hierarchy to include negative flows. This construction establishes a purely algebraic link between, on the one hand, two realizations of the first negative flow of the AKNS model and, on the other, two-component generalizations of Camassa–Holm- and Dym-type equations. The two-component generalizations of Camassa–Holm- and Dym-type equations can be obtained from the negative-order Hamiltonians constructed from the Lenard relations recursively applied on the Casimir of the first Poisson bracket of hydrodynamic type. The positive-order Hamiltonians, which follow from the Lenard scheme applied on the Casimir of the second Poisson bracket of hydrodynamic type, are shown to coincide with the Hamiltonians of the AKNS model. The AKNS Hamiltonians give rise to charges conserved with respect to equations of motion of two-component Camassa–Holm- and two-component Dym-type equations.

PACS number: 02.30.Ik

1. Introduction

Recently, the celebrated shallow-water equation obtained by Camassa and Holm [1]

$$u_t - u_{txx} = -3uu_x + 2u_xu_{xx} + uu_{xxx} - \kappa u_x, \qquad \kappa = \text{const}, \tag{1}$$

was extended in [2] by adding on the right-hand side a term $\rho\rho_x$ with a new variable ρ, which satisfies the continuity equation $\rho_t + (u\rho)_x = 0$. The model resulting from the above generalization first appeared in the study of deformations of the bihamiltonian structure of hydrodynamic type [3, 4] and was named the two-component Camassa–Holm equation. Soon after its derivation the model was identified with the first negative flow of the AKNS hierarchy [2].

Another well-known integrable partial differential equation of interest to our study is the Dym-type equation [5–10]:

$$-u_{xxt} = 2u_x u_{xx} + u u_{xxx} - \kappa u_x. \tag{2}$$

It can also be extended to two-component version by adding a term $\rho\rho_x$ on the right-hand side of (2). The resulting two variable system is shown here to be equivalent to the negative flow of one of the extensions of the AKNS model. It is also equivalent to the special limiting procedure of deformations of the bihamiltonian structure of hydrodynamic type.

In this paper the following is accomplished: first, we explore the Schrödinger spectral problem of second order describing both two-component Camassa–Holm- and Dym-type equations for different values of the deformation parameter μ. We show that this Schrödinger spectral problem can be cast into the linear 2×2 matrix spectral problem. Using $sl(2)$ gauge invariance we transform the time evolution flow of the linear spectral problem into the AKNS first negative flow. We should point out that there exist several ways to extend the AKNS hierarchy to incorporate negative flows. These extensions are parametrized by a single parameter identified with μ. We associate two different constructions of the negative flows of the AKNS hierarchy with the two-component Camassa–Holm- (for $\mu = 1$) and Dym-type equations (for $\mu = 0$). The result concerning the two-component Camassa–Holm equation constitutes an algebraic version of the proof given in [2]. Using connections of the AKNS and deformed Sinh-Gordon models to the two-component Camassa–Holm- and Dym-type equations, respectively, we are able to find explicit soliton solutions given in hodographic variables. The relation to the AKNS models allows us to construct a new chain of charges conserved with respect to equations of motion of two-component Camassa–Holm- and Dym-type equations. For both hierarchies the modified AKNS Hamiltonians provide a tower of positive-order Hamiltonians obtained via the underlying Lenard relations of the Poisson brackets of hydrodynamic type from the Casimir of the second bracket.

In section 2, we briefly review the algebraic approach to the AKNS model and show how to extend the model in two different ways to negative time flows based on the zero-curvature identities. In section 3, we set up a class of two-component Schrödinger spectral problems parametrized by μ. In section 4, we transform the Schrödinger spectral problem by the reciprocal transformation and linearize it. The resulting linear 2×2 matrix spectral problem is then transformed by an $sl(2)$ gauge transformation into the AKNS Lax spectral problem. The time flows of two-component Camassa–Holm- and Dym-type equations are shown to coincide with two different negative flows of the extended AKNS model. Our construction allows us to find, in subsection 4.1, explicit soliton solutions for various values of μ. In section 5, we reproduce equations of motion for $\mu \neq 0$ and $\mu = 0$ cases in the setting of deformations of the bihamiltonian structure of hydrodynamic type. Remarkably, the Hamiltonians governing positive evolution flows of the AKNS hierarchy define conserved charges for the two-component Camassa–Holm- and Dym-type equations. Also the conserved charges induced by the AKNS model satisfy among themselves the Lenard relations of the bihamiltonian structure of hydrodynamic type. Thus, the Hamiltonians of the bihamiltonian structure of hydrodynamic type connected to two-component Camassa–Holm- and Dym-type equations split into two chains, one of the positive order induced by the AKNS hierarchy and the other of the negative order containing generators of the equations of motion defining both hierarchies.

2. Extended AKNS model

First, let us present the AKNS hierarchy in the setting of the $sl(2)$ loop algebra endowed with homogeneous gradation defined by the operator $\lambda\, \mathrm{d}/\mathrm{d}\lambda$. A variable λ plays a double role of a loop parameter of the loop algebra and a spectral parameter of the underlying hierarchy. The matrix Lax operator L for the AKNS hierarchy reads

$$L = \frac{\partial}{\partial y} - \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} - \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}, \tag{3}$$

where $\partial/\partial y$ is the derivative with respect to 'space' variable y. The matrix Lax operator can be compactly written as $L = \partial/\partial y - E - A_0$, with $E = \lambda\sigma_3$ and the matrix $A_0 = q\sigma_+ + r\sigma_-$, where σ_3 is the Pauli matrix and $\sigma_\pm$ are given in terms of other Pauli matrices σ_1, σ_2:

$$\sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We work within an algebraic approach to the integrable models based on the linear spectral problem $L(\Psi) = 0$, which simplifies considerably under a dressing transformation:

$$\Theta^{-1}\left(\frac{\partial}{\partial y} - E - A_0\right)\Theta = \frac{\partial}{\partial y} - E, \tag{4}$$

where the dressing matrix $\Theta = \exp\left(\sum_{i<0} \lambda^i \theta^{(i)}\right)$ is an exponential in negative powers of the spectral parameter λ on a formal loop space of $sl(2)$. Similarly, for higher flows we obtain

$$\Theta^{-1}\left(\frac{\partial}{\partial t_n} - E^{(n)} - \sum_{i=0}^{n-1} \lambda^i D_n^{(i)}\right)\Theta = \frac{\partial}{\partial t_n} - E^{(n)}, \tag{5}$$

where $E^{(n)} = \lambda^n \sigma_3$ and terms $D_n^{(i)}$ are obtained from the projection $(\Theta E^{(n)} \Theta^{-1})_+$ of $\Theta E^{(n)} \Theta^{-1}$ on the positive powers of λ via expansion relation

$$(\Theta E^{(n)} \Theta^{-1})_+ = E^{(n)} + \sum_{i=0}^{n-1} \lambda^i D_n^{(i)}.$$

These dressing relations give rise to the zero-curvature conditions for the positive flows of the AKNS hierarchy:

$$\left[\frac{\partial}{\partial y} - E - A_0, \frac{\partial}{\partial t_n} - E^{(n)} - \sum_{i=0}^{n-1} D_n^{(i)}\right] = \Theta\left[\frac{\partial}{\partial y} - E, \frac{\partial}{\partial t_n} - E^{(n)}\right]\Theta^{-1} = 0. \tag{6}$$

In particular, for $n = 2$ we obtain the second flow of the AKNS hierarchy:

$$\frac{\partial r}{\partial t_2} = -\frac{1}{2} r_{yy} + q r^2; \qquad \frac{\partial q}{\partial t_2} = \frac{1}{2} q_{yy} - q^2 r, \tag{7}$$

which reproduces the familiar vector nonlinear Schrödinger equation.

According to [11], the Hamiltonian densities of the AKNS model are defined as

$$\mathcal{H}_n = -\mathrm{tr}(E^{(0)} A^{(-n)}) = \frac{1}{2} \sum_{k=0}^{n-1} \mathrm{tr}(A^{(-k)} A^{(1+k-n)}), \tag{8}$$

where $A^{(-n)}$ are given by

$$\Theta_y \Theta^{-1} = \sum_{k=1}^{\infty} A^{(-k)} \lambda^{-k},$$

where the symbol tr in expression (8) denotes a $sl(2)$ trace. We list the first two Hamiltonians. Inserting $n = 1$ in (8) we obtain

$$\mathcal{H}_1 = -\mathrm{tr}(E^{(0)} A^{(-1)}) = \tfrac{1}{2}\,\mathrm{tr}\big(A_0^2\big) = rq. \tag{9}$$

Similarly, for $n = 2$ we obtain

$$\mathcal{H}_2 = q r_y. \tag{10}$$

Next, we extend the AKNS model to include negative grade time evolution equations governed by the zero-curvature equations [12]

$$\left[\frac{\partial}{\partial y} - E - A_0, \frac{\partial}{\partial t_{-n}} - D^{(-1)} - D^{(-2)} - \cdots - D^{(-n)}\right] = 0. \tag{11}$$

Here, we only consider the first negative flow with $n = 1$ and set for brevity $s = t_{-1}$. In this case, the compatibility equation (11) reduces to

$$(A_0)_s - D^{(-1)}_y + [E + A_0, D^{(-1)}] = 0. \tag{12}$$

A general solution of the compatibility equation (12) is given by

$$D^{(-1)} = B\mathcal{E}^{(-1)}B^{-1}, \qquad A_0 = B_y B^{-1}, \tag{13}$$

in terms of the zero-grade group element, B, of $SL(2)$, that satisfies equation

$$(B_y B^{-1})_s = [B\mathcal{E}^{(-1)}B^{-1}, E] \tag{14}$$

or, equivalently,

$$(B^{-1}B_s)_y = [\mathcal{E}^{(-1)}, B^{-1}EB]. \tag{15}$$

Here $\mathcal{E}^{(-1)}$ is an element of $sl(2)$ algebra of -1 grade.

Remarkably, the compatibility of the t_{-1} flow with positive $t_n, n \geqslant 1$, flows does not require that the matrix $\mathcal{E}^{(-1)}$ commutes with $E = \lambda\sigma_3$, as pointed out in [13] and [14]. It turns out that all possible cases are parametrized by a parameter μ and fall into two main classes depending on whether μ takes non-zero or zero value. The corresponding generic choices of $\mathcal{E}^{(-1)}$ are

$$\mathcal{E}^{(-1)} = \begin{cases} \mu\sigma_3/4\lambda & \text{for} \quad \mu \neq 0 \\ \sigma_+/\lambda & \text{for} \quad \mu = 0. \end{cases} \tag{16}$$

Note that the value of the determinant of $\mathcal{E}^{(-1)}$ is equal to $-\mu^2/16\lambda^2$ and 0, respectively. There exist other choices of $\mathcal{E}^{(-1)}$ for these values of the determinant but they only lead to the gauge-equivalent copies of hierarchies derived from choice (16).

3. A class of two-component Schrödinger spectral problems

Consider a linear system

$$\psi_{xx} = \left(\frac{\mu^2}{4} - m\lambda + \rho^2\lambda^2\right)\psi, \tag{17}$$

$$\psi_t = -\left(\frac{1}{2\lambda} + u\right)\psi_x + \frac{1}{2}u_x\psi, \tag{18}$$

for some arbitrary constant μ (see [2, 3] for $\mu = 1$ and [15] for $\mu = 0$). Compatibility condition for the above system yields three independent equations

$$\rho_t = -(u\rho)_x, \tag{19}$$

$$m_t = -2mu_x - m_x u + \rho\rho_x, \tag{20}$$

$$m_x = \mu^2 u_x - u_{xxx}, \tag{21}$$

corresponding to coefficients of λ^2, λ and λ^0 in the expansion of $\psi_{xxt} - \psi_{txx} = 0$. Equation (21) can be integrated to yield

$$m = \mu^2 u - u_{xx} + \tfrac{1}{2}\kappa. \tag{22}$$

Here, κ is an integration constant. For $\mu \neq 0$ that integration constant can be removed by transforming the system by Galilean transformation:

$$x' = x + vt, \qquad t' = t, \qquad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial t} = v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}.$$

In the primed system equation (20) becomes

$$v m_{x'} + m_{t'} = -2mu_{x'} - m_{x'}u + \rho\rho_{x'} = -2\big(\mu^2 u - u_{x'x'} + \tfrac{1}{2}\kappa\big)u_{x'} - m_{x'}u + \rho\rho_{x'}.$$

Next, performing a shift $u \to u - v$ and choosing velocity v such that $v = \kappa/2\mu^2$ eliminates the linear terms in u_x and u_{xxx} from the above equation. Clearly, the above argument works only for $\mu \neq 0$ and from now on we put the integration constant κ to zero as long as $\mu \neq 0$.

Note that the positive constant μ^2 that is different from 1 can be absorbed by appropriately redefining fields and derivatives. Defining $\widetilde{\lambda} = \lambda/\mu^2, \widetilde{\rho} = \rho\mu, \widetilde{u} = u\mu^2$ and new variables $\widetilde{x}$ and $\widetilde{t}$ such that $\partial_x = \mu\partial_{\widetilde{x}}, \partial_t = (1/\mu)\partial_{\widetilde{t}}$ allows us to rewrite a linear system (17), (18) as

$$\psi_{\tilde{x}\tilde{x}} = \left(\frac{1}{4} - m\widetilde{\lambda} + \widetilde{\rho}^2\widetilde{\lambda}^2\right)\psi, \tag{23}$$

$$\psi_{\tilde{t}} = -\left(\frac{1}{2\widetilde{\lambda}} + \tilde{u}\right)\psi_{\tilde{x}} + \frac{1}{2}\widetilde{u}_{\tilde{x}}\psi \tag{24}$$

with $m = \mu^2 u - u_{xx} + \kappa/2 = \widetilde{u} - \widetilde{u}_{\tilde{x}\tilde{x}} + \kappa/2$. Thus, for $\mu^2 \neq 1$ the spectral system has been transformed to the canonical system with $\mu^2 = 1$.

In the case of a negative μ^2 (imaginary μ), we make the changes as above but with $|\mu^2|$ instead of μ^2 and arrive at

$$\psi_{\tilde{x}\tilde{x}} = \big(-\tfrac{1}{4} - m\widetilde{\lambda} + \widetilde{\rho}^2\widetilde{\lambda}^2\big)\psi. \tag{25}$$

Thus, only three cases of $\mu^2 = -1, 0, 1$ need to be considered separately, as concerns equations of motion.

The case of $\mu^2 = 1$ corresponds to the two-component Camassa–Holm model with $m = u - u_{xx} + \kappa/2$, introduced in [2, 3], while the case of $\mu^2 = -1$ corresponds to equation (20) with $m = -u - u_{xx} + \kappa/2$, which for $\rho = 0$ is well known to possess the compacton solutions [16].

For $\mu = 0$, we obtain from (22) $m = -u_{xx} + \kappa/2$. Inserting this value of m into equation (20) yields

$$u_{xxt} = -2u_x u_{xx} - uu_{xxx} + \kappa u_x - \rho\rho_x. \tag{26}$$

For $\rho = 0$, this is the Dym-type equation (2).

After one integration (and ignoring the integration constant) we obtain from (26)

$$\begin{aligned} 0 &= u_{xt} + uu_{xx} - \kappa u + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}\rho^2 \\ &= (u_t + uu_x)_x - \kappa u + \tfrac{1}{2}\left(-u_x^2 + \rho^2\right). \end{aligned} \tag{27}$$

In terms of a new function

$$v = \tfrac{1}{2}\left(-u_x^2 + \rho^2\right), \tag{28}$$

we can cast equation (27) in the following form:

$$(u_t + uu_x)_x - \kappa u + v = 0. \tag{29}$$

In addition, it follows from equations (27) and (19) that v defined by relation (28) also satisfies

$$v_t + (u(v + u\kappa/2))_x = 0, \tag{30}$$

which becomes a continuity equation in the $\kappa = 0$ limit. The linear system corresponding to equations (29) and (30) takes a form

$$\begin{aligned} \psi_{xx} &= \left((u_{xx} - \kappa/2)\lambda + \left(2v + u_x^2\right)\lambda^2\right)\psi, \\ \psi_t &= -\left(\frac{1}{2\lambda} + u\right)\psi_x + \frac{1}{2}u_x\psi. \end{aligned} \tag{31}$$

4. Transformation to the first-order spectral problem. Algebraic connection to the AKNS model

Now, for an arbitrary μ we perform a reciprocal transformation $(x, t) \mapsto (y, s)$ defined by relations

$$\mathrm{d}y = \rho\,\mathrm{d}x - \rho u\,\mathrm{d}t, \qquad \mathrm{d}s = \mathrm{d}t, \tag{32}$$

and

$$\frac{\partial}{\partial x} = \rho\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} - \rho u\frac{\partial}{\partial y}. \tag{33}$$

The commutativity of derivatives with respect to s and y variables is ensured by the continuity equation (19). Applying the reciprocal transformation and then redefining ψ by $\varphi = \sqrt{\rho}\psi$ as in [2] lead from the spectral problem (17), (18) to

$$\varphi_{yy} = (\lambda^2 - P\lambda - Q)\varphi, \tag{34}$$

$$\varphi_s = -\frac{\rho}{2\lambda}\varphi_y + \frac{\rho_y}{4\lambda}\varphi, \tag{35}$$

where

$$P = \frac{m}{\rho^2} \qquad Q = -\frac{\mu^2}{4\rho^2} - \frac{\rho_{yy}}{2\rho} + \frac{\rho_y^2}{4\rho^2}. \tag{36}$$

Our main point in this section is that we can rewrite the second-order spectral problem (34), (35) as a first-order linear problem:

$$\begin{bmatrix} \varphi \\ \eta \end{bmatrix}_y = A\begin{bmatrix} \varphi \\ \eta \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} \varphi \\ \eta \end{bmatrix}_s = D\begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \tag{38}$$

involving $sl(2)$ matrices:

$$\begin{aligned} A &= \begin{bmatrix} g & \lambda \\ \lambda - P & -g \end{bmatrix} = \lambda\sigma_1 + g\sigma_3 - P\sigma_-, \\ D &= \frac{1}{\lambda}D_0 - \frac{1}{2}\rho\sigma_1, \qquad D_0 = -\frac{\mu}{4}\sigma_3 + \frac{1}{4}\left(P\rho - 2g_s\right)\sigma_-. \end{aligned} \tag{39}$$

Note that determinant of D_0 is equal to $\det D_0 = -\mu^2/16$ and, therefore, the matrix D_0 becomes singular for $\mu = 0$. Eliminating η from the linear spectral problem (37), (38) reproduces equations (34) and (35) for φ provided that function $g(y, s)$ appearing in (39) satisfies the Riccati equation

$$Q = -g^2 - g_y,$$

for Q given in equation (36). Remarkably, the solution to the above Riccati equation takes a local form of

$$g(y,s) = \frac{\mu}{2\rho} + \frac{\rho_y}{2\rho}. \tag{40}$$

The zero-curvature equation

$$A_s - D_y + [A, D] = 0, \tag{41}$$

can easily be derived from the linear spectral problem (37), (38). It is equivalent to equations

$$P_s = \rho_y, \qquad Q_s + \tfrac{1}{2} P_y \rho + P \rho_y = 0. \tag{42}$$

These equations were found in [2] directly from compatibility of equations (34) and (35). It follows from the first of the above equations that there exists a function $f(y,s)$ such that $P = f_y$ and $\rho = f_s$.

By plugging $\rho = f_s$ and $P = f_y$ into the second relation in (42) one obtains as in [2]:

$$\mu^2 \frac{f_{ss}}{2 f_s^3} + f_{sy} f_y + \frac{1}{2} f_s f_{yy} - \frac{f_{ssyy}}{2 f_s} + \frac{f_{ssy} f_{sy}}{2 f_s^2} + \frac{f_{ss} f_{syy}}{2 f_s^2} - \frac{f_{ss} f_{sy}^2}{2 f_s^3} = 0. \tag{43}$$

This appears to be the only condition, which the function f has to satisfy in order to be a solution of the model.

For $\mu = 0$ equation (43) simplifies to

$$\left(f_s^2 f_y - f_{ssy} + \frac{f_{ss} f_{sy}}{f_s} \right)_y = 0. \tag{44}$$

Integrating the above equation once and setting the integration constant to $\kappa/2$ (see the explanation below) yield

$$\frac{f_{ssy}}{f_s^2} - \frac{f_{ss} f_{sy}}{f_s^3} + \frac{\frac{1}{2}\kappa}{f_s^2} = f_y, \tag{45}$$

or

$$(\ln f_s)_{sy} + \tfrac{1}{2}\kappa / f_s = f_s f_y.$$

Indeed, multiplying both sides of equation (45) by f_s^2 and taking a derivative with respect to y yield (44). It remains to be shown that the choice of $\kappa/2$ as the integration constant in equation (45) was consistent with equations of motion. To do this we start by recalling that $P = m/\rho^2$ with

$$m = -u_{xx} + \tfrac{1}{2}\kappa = -\rho(\rho u_y)_y + \tfrac{1}{2}\kappa$$

in the $\mu = 0$ case. The continuity equation (19) reads in the hodographic variables $\rho_s = -\rho^2 u_y$. Accordingly, substituting $u_y = (1/\rho)_s$ into P we get

$$P = \frac{(\rho_s/\rho)_y}{\rho} + \frac{\frac{1}{2}\kappa}{\rho^2} = f_y,$$

which is precisely equation (45).

Let us turn our attention back to the zero-curvature equation (41). This equation is invariant under the $sl(2)$ gauge transformation:

$$A \longrightarrow U A U^{-1} + U_y U^{-1}, \qquad D \longrightarrow U D U^{-1} + U_s U^{-1}.$$

This invariance will be used in what follows to cast the linear spectral problem (37), (38) in the standard form of the first positive and first negative flow equations of the $sl(2)$ AKNS hierarchy.

As a first step we gauge away the term $-\frac{1}{2}\rho\sigma_1$ of order λ^0 in the expression for D in equation (39) by choosing

$$U = \exp\left(\frac{1}{2} f(y,s)\sigma_1\right) = \cosh\frac{f}{2} + \sigma_1 \sinh\frac{f}{2},$$

so that $U_s U^{-1} - \frac{1}{2}\rho\sigma_1 = 0$, due to $f_s = \rho$. Consequently,

$$\begin{aligned} A \to UAU^{-1} + U_y U^{-1} &= U(\lambda\sigma_1 + g\sigma_3 - P\sigma_-)U^{-1} + \frac{1}{2}P\sigma_1 \\ &= \lambda\sigma_1 + \sigma_3\left(g\cosh f - \frac{1}{2}P\sinh f\right) - \mathrm{i}\sigma_2\left(g\sinh f - \frac{1}{2}P\cosh f\right) \end{aligned}$$

$$\begin{aligned} D \to \frac{1}{\lambda}UD_0U^{-1} = \frac{1}{4\lambda}[&(-\sigma_1\cosh f - \mathrm{i}\sigma_2\sinh f)\mu \\ &+ (P\rho - 2g_s)(\sigma_3 + \sigma_1\cosh f + \mathrm{i}\sigma_2\sinh f)]. \end{aligned}$$

Note that the gauge transformed of the matrix D is now proportional to $1/\lambda$.

Next, we define the constant matrix $\Omega = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)$, that by a similarity transformation maps σ_1 to σ_3, $\Omega\sigma_1\Omega^{-1} = -\sigma_3$. Also note that $\Omega\sigma_2\Omega^{-1} = -\sigma_2$. The combined gauge transformations first by U and then by Ω produce the final result

$$\begin{aligned} A \to E + A_0 &= \Omega[UAU^{-1} + U_yU^{-1}]\Omega^{-1} \\ &= \lambda\sigma_3 + \sigma_1\left(g\cosh f - \frac{1}{2}P\sinh f\right) + \mathrm{i}\sigma_2\left(g\sinh f - \frac{1}{2}P\cosh f\right) \\ D \to D^{(-1)} &= \frac{1}{\lambda}\Omega U D_0 U^{-1}\Omega^{-1} = \frac{1}{4\lambda}[(P\rho - 2g_s)\sigma_3 + \sigma_1((P\rho - 2g_s)\sinh f \\ &\quad - \mu\cosh f) + \mathrm{i}\sigma_2((P\rho - 2g_s)\cosh f - \mu\sinh f)]. \end{aligned} \tag{46}$$

In the above, we re-introduced $E = \lambda\sigma_3$ and $A_0 = r\sigma_- + q\sigma_+$. Compared with the right-hand side of equation (46) we find that

$$q = \mathrm{e}^f\left(g - \frac{P}{2}\right) \qquad r = \mathrm{e}^{-f}\left(g + \frac{P}{2}\right). \tag{47}$$

Furthermore, defining matrix entries of $D^{(-1)}$ as

$$D^{(-1)} = \frac{1}{\lambda}\begin{bmatrix}\alpha & \beta \\ \gamma & -\alpha\end{bmatrix}, \tag{48}$$

we find from (46) that α, β and γ are given by

$$\alpha = \frac{1}{4}(P\rho - 2g_s) \qquad \beta = \mathrm{e}^f\left(\alpha - \frac{\mu}{4}\right) \qquad \gamma = -\mathrm{e}^{-f}\left(\alpha + \frac{\mu}{4}\right). \tag{49}$$

They satisfy the determinant formula $\alpha^2 + \beta\gamma = -\mu^2/16$. The matrix entries of A_0 and $D^{(-1)}$ enter the following simple identities:

$$2\alpha = \beta\,\mathrm{e}^{-f} - \gamma\,\mathrm{e}^f, \qquad -\frac{\mu}{2} = \beta\,\mathrm{e}^{-f} + \gamma\,\mathrm{e}^f \tag{50}$$

$$P = r\,\mathrm{e}^f - q\,\mathrm{e}^{-f}, \qquad g = \frac{1}{2}(r\,\mathrm{e}^f + q\,\mathrm{e}^{-f}). \tag{51}$$

It follows that the linear spectral problem (37), (38) has been transformed by the above gauge transformation to

$$\Psi_y = (E + A_0)\,\Psi = \begin{bmatrix}\lambda & 0 \\ 0 & -\lambda\end{bmatrix}\Psi + \begin{bmatrix}0 & q \\ r & 0\end{bmatrix}\Psi \tag{52}$$

$$\Psi_s = D^{(-1)}\Psi = \frac{1}{\lambda}\begin{bmatrix}\alpha & \beta \\ \gamma & -\alpha\end{bmatrix}\Psi \tag{53}$$

for some two-component object Ψ.

We recognize in (52) the spectral problem $L(\Psi) = 0$ with the AKNS Lax operator given by equation (3). It also follows easily that equation (12) is the compatibility equation of the spectral equations (52) and (53). The compatibility equation (12) yields

$$q_s = -2\beta, \qquad r_s = 2\gamma, \tag{54}$$

when projected on zero grade, and

$$\alpha_y = \tfrac{1}{2}(rq)_s = q\gamma - r\beta \qquad \beta_y = -2\alpha q \qquad \gamma_y = 2\alpha r, \tag{55}$$

when projected on -1 grade. Equations (55) can also be directly derived from equations of motion (42).

4.1. Examples and solutions

Let us recall that a general solution of the compatibility equation (12) is given by expressions from equation (13). It is convenient to parametrize the $SL(2)$ group element B appearing in expressions (13) by the $sl(2)$ algebra elements through the Gauss decomposition:

$$B = \mathrm{e}^{\chi\sigma_-}\,\mathrm{e}^{R\sigma_3}\,\mathrm{e}^{\phi\sigma_+}. \tag{56}$$

4.1.1. The case of $\mu \neq 0$. As an example, we first consider $\mu^2 = 4$ with $\mathcal{E}^{(-1)} = \sigma_3/2\lambda$ according to equation (16). As in [12] in order to match the number of independent modes in the matrix A_0 we impose two 'diagonal' constraints $\mathrm{Tr}(B_y B^{-1}\sigma_3) = 0$ and $\mathrm{Tr}(B^{-1}B_s\sigma_3) = 0$, which effectively eliminate R in terms of ϕ and χ.

From $B_y B^{-1} = q\sigma_+ + r\sigma_-$ we obtain the following representation for q and r:

$$q = -\frac{h_y}{\Delta}\,\mathrm{e}^R; \qquad r = -\bar{h}_y\,\mathrm{e}^{-R}, \tag{57}$$

where

$$h = \phi\,\mathrm{e}^R; \qquad \bar{h} = \chi\,\mathrm{e}^R; \qquad \Delta = 1 + h\bar{h} \tag{58}$$

with a non-local field R being determined in terms h and $\bar{h}$ from the 'diagonal' constraints:

$$\mathrm{Tr}(B_y B^{-1}\sigma_3) = 0 \to R_y = \frac{\bar{h}h_y}{\Delta}, \tag{59}$$

$$\mathrm{Tr}(B^{-1}B_s\sigma_3) = 0 \to R_s = \frac{h\bar{h}_s}{\Delta}. \tag{60}$$

The zero-curvature equations are in this parametrization

$$q_s = \left(-\frac{h_y}{\Delta}\,\mathrm{e}^R\right)_s = -2h\,\mathrm{e}^R, \tag{61}$$

$$r_s = (-\bar{h}_y\,\mathrm{e}^{-R})_s = -2\bar{h}\Delta\,\mathrm{e}^{-R}. \tag{62}$$

The two-parameter solution to the above equations can be deduced from a method combining dressing and vertex techniques as described e.g. in [17]. The explicit expression is found to be given by

$$h = \frac{b\exp(s/\gamma_2 - y\gamma_2)}{1+\Gamma\exp\left(s\left(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\right)-y(\gamma_2-\gamma_1)\right)}$$
$$\bar h = \frac{a\exp(-s/\gamma_1 + y\gamma_1)}{1+\Gamma\exp\left(s\left(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\right)-y(\gamma_2-\gamma_1)\right)} \tag{63}$$
$$\mathrm{e}^R = \frac{1+\frac{\gamma_1}{\gamma_2}\Gamma\exp\left(s\left(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\right)-y(\gamma_2-\gamma_1)\right)}{1+\Gamma\exp\left(s\left(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\right)-y(\gamma_2-\gamma_1)\right)},$$

where

$$\Gamma = \frac{ab\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2}$$

is given in terms of four arbitrary constants a, b, γ_1, γ_2. Higher multi-soliton solutions can be obtained using the same straightforward procedure.

Compared with equations (54) we find that

$$h\,\mathrm{e}^R = \mathrm{e}^f\alpha_-, \qquad \bar h\Delta\,\mathrm{e}^{-R} = \mathrm{e}^{-f}\alpha_+, \tag{64}$$

where we introduced the notation

$$\alpha_\pm = \alpha \pm \tfrac{1}{2}.$$

By multiplying the above two relations we find that

$$h\bar h\Delta = (\Delta-1)\,\Delta = \alpha_+\alpha_- = \alpha^2 - \tfrac{1}{4}.$$

Solutions to this quadratic equation are

$$\Delta = \pm\alpha_\pm, \qquad h\bar h = \Delta - 1 = \pm\alpha_\mp. \tag{65}$$

Adding two relations in (64) we get

$$2\alpha = \alpha_+ + \alpha_- = hx + \frac{1}{x}\bar h\Delta, \qquad x = \mathrm{e}^{R-f}.$$

Solving this quadratic equation yields

$$f_\pm = R + \ln h - \ln\alpha_\pm. \tag{66}$$

Equation (65) contains two solutions. The first one, namely, $\alpha_+ = \Delta, \alpha_- = \Delta - 1$, when inserted into equation (66) yields

$$f_+ = R + \ln h - \ln\Delta, \qquad f_- = R + \ln h - \ln(\Delta-1) = R - \ln\bar h,$$

while the second solution, $\alpha_+ = -(\Delta-1), \alpha_- = -\Delta$, leads to

$$f_+ = R + \ln h - \ln(-\Delta+1), \qquad f_- = R + \ln h - \ln(-\Delta).$$

Thus, the Bäcklund transformation

$$f_+ = f_- + \epsilon\ln\left(\frac{\Delta}{\Delta-1}\right), \qquad \epsilon = \pm 1$$

relates the two values f_+ and f_-. In the reduced case of sinh-Gordon equation with $h = \bar h$ we find that $\Delta = 1 + h^2$ and

$$R = \tfrac{1}{2}\ln(1+h^2) = \tfrac{1}{2}\ln\Delta, \qquad \ln(\Delta-1) = 2\ln h.$$

It follows that all the above values of f_+, f_- can be summarized as

$$f_\epsilon = \frac{\epsilon}{2}\ln\frac{1+h^2}{h^2}, \qquad \epsilon = \pm 1.$$

4.1.2. The case of $\mu = 0$. In the case of $\mu = 0$, the matrix $D^{(-1)}$ from equation (48) takes a simple form

$$D^{(-1)} = \frac{\alpha}{\lambda} \begin{bmatrix} 1 & e^{f} \\ -e^{-f} & -1 \end{bmatrix}, \tag{67}$$

which according to definition (16) is reproduced by

$$\frac{1}{\lambda} B\sigma_+ B^{-1} = \frac{1}{\lambda} \begin{bmatrix} -\chi & 1 \\ -\chi^2 & \chi \end{bmatrix} e^{2R}$$

for

$$\chi = -e^{-f} \qquad e^{2R} = \alpha\, e^{f} \tag{68}$$

or $R = (f + \ln\alpha)/2$.

From equation (55) we find for $\mu = 0$ that $\alpha_y = -2\alpha g$. Therefore,

$$g = -\frac{\alpha_y}{2\alpha} = -\frac{1}{2}(\ln\alpha)_y \tag{69}$$

and by comparing with definition (40) we conclude that

$$(\rho\alpha)_y = 0. \tag{70}$$

Next, we calculate

$$B_y B^{-1} = (R_y - \chi\phi_y\, e^{2R})\sigma_3 + \phi_y\, e^{2R}\sigma_+ + (\chi_y + 2\chi R_y - \chi^2\phi_y\, e^{2R})\sigma_-. \tag{71}$$

Imposing condition $\mathrm{Tr}(B_y B^{-1}\sigma_3) = 0$ implies that $R_y - \chi\phi_y\, e^{2R} = 0$ or

$$\phi_y = -R_y/\alpha = -\frac{1}{2\alpha}\left(f_y + \frac{\alpha_y}{\alpha}\right) = \frac{1}{\alpha}(g - P/2).$$

What remains of expression (71) is now given by

$$B_y B^{-1} = \phi_y\, e^{2R}\sigma_+ + (\chi_y + \chi R_y)\sigma_- = (R_y/\chi)\sigma_+ + (\chi_y + \chi R_y)\sigma_-.$$

Recalling relations (68) and (69) we obtain the desired results

$$q = \phi_y\, e^{2R} = \left(g - \tfrac{1}{2}P\right) e^{f}, \qquad r = \chi_y + \chi R_y = \left(g + \tfrac{1}{2}P\right) e^{-f},$$

that reproduce expressions (47).

The compatibility equation

$$(B_y B^{-1})_s = [B\sigma_+ B^{-1}, \sigma_3] = \begin{bmatrix} 0 & -2 \\ -2\chi^2 & 0 \end{bmatrix} e^{2R}$$

yields the following equations of motion:

$$\left(\frac{R_y}{\chi}\right)_s = -2e^{2R} \tag{72}$$

$$\chi_{ys} + 2\chi_s R_y = 0 \to (\chi_s\, e^{2R})_y = 0. \tag{73}$$

Equation (73) implies that

$$\chi_s = c_3(s)\, e^{-2R}, \tag{74}$$

where $c_3(s)$ is an arbitrary function of s only.

From equations (68) we find that

$$\chi_s = f_s\, e^{-f} = \rho\, e^{-f} = c_3(s)\alpha^{-1}\, e^{-f}$$

and therefore $\rho = c_3(s)/\alpha$ in agreement with relation (70). It follows that $c_3(s)$ has to be different from zero for consistency of the model with $\rho \neq 0$.

Integrating relations (74) and (72) leads to

$$\chi = \int^{s} c_3(s)\, \mathrm{e}^{-2R}\, \mathrm{d}s + c_2(y) \tag{75}$$

$$\frac{R_y}{\chi} = -2 \int^{s} \mathrm{e}^{2R}\, \mathrm{d}s + c_1(y), \tag{76}$$

where c_1, c_2 depend at most on y. Combining these two equations and setting $c_3(s)$ to be a constant c_3 we get the deformed sinh-Gordon equation for R [18]:

$$R_y = -2c_3 \int^{s} \mathrm{e}^{-2R}\, \mathrm{d}s \int^{s} \mathrm{e}^{2R}\, \mathrm{d}s - 2c_2 \int^{s} \mathrm{e}^{2R}\, \mathrm{d}s + c_1 c_3 \int^{s} \mathrm{e}^{-2R}\, \mathrm{d}s + c_1 c_2 \tag{77}$$

or

$$R_{ys} = -2c_3 \left(\int^{s} \mathrm{e}^{-2R}\, \mathrm{d}s \int^{s} \mathrm{e}^{2R}\, \mathrm{d}s \right)_s - 2c_2\, \mathrm{e}^{2R} + c_1 c_3\, \mathrm{e}^{-2R}.$$

The one-soliton solution to the above equation with $c_1 = c_2 = 0$ is given by (see also [18])

$$R(s, y) = \frac{s}{2} + 2c_3 y + \ln \left(\frac{k_0\, \mathrm{e}^{k_1 s + k_2 y} + k_1 + 1}{k_0\, \mathrm{e}^{k_1 s + k_2 y} - k_1 + 1} \right), \tag{78}$$

where

$$k_2 = 8c_3 \frac{k_1}{k_1^2 - 1}$$

and k_0, k_1 are real constants. The corresponding expression for χ is

$$\chi(s, y) = -\mathrm{e}^{-f} = -\mathrm{e}^{-s - 4c_3 y} \frac{(k_1 - 1)^2/(k_1 + 1) + k_0\, \mathrm{e}^{k_1 s + k_2 y}}{k_0\, \mathrm{e}^{k_1 s + k_2 y} + k_1 + 1} c_3. \tag{79}$$

The above function χ together with R from (78) solves equations (72) and (73). The function f defined by equation (79) provides a one-soliton solution to equation (44). It satisfies

$$f_s = \frac{c_3}{\alpha}$$

with

$$\alpha = \frac{((k_1 - 1)^2/(k_1 + 1) + k_0\, \mathrm{e}^{k_1 s + k_2 y})(k_0\, \mathrm{e}^{k_1 s + k_2 y} + k_1 + 1)}{(k_0\, \mathrm{e}^{k_1 s + k_2 y} - k_1 + 1)^2} c_3. \tag{80}$$

Eliminating α from equation (72) using (68) we get

$$-(\ln f_s)_{ys} + f_s f_y = 4\alpha = \frac{4c_3}{f_s}.$$

Therefore, comparing with (45), we see that $\kappa = 8c_3$ and f given in (79) satisfies (45) and therefore also (44). The one-soliton solution $R(s, y)$ given by expression (78) satisfies therefore

$$R_y = -\frac{\kappa}{4} \int^{s} \mathrm{e}^{-2R}\, \mathrm{d}s \int^{s} \mathrm{e}^{2R}\, \mathrm{d}s.$$

5. The bihamiltonian structure

5.1. The bihamiltonian structure of the two-component Camassa–Holm model

As in [3], we consider the following biHamiltonian structure:

$$\begin{aligned}
&\{w_1(x), w_1(x')\}_1 = \{w_2(x), w_2(x')\}_1 = 0,\\
&\{w_1(x), w_2(x')\}_1 = \delta'(x-x') - \frac{1}{\mu}\delta''(x-x').\\
&\{w_1(x), w_1(x')\}_2 = 2\delta'(x-x'),\\
&\{w_1(x), w_2(x')\}_2 = w_1(x)\delta'(x-x') + w_1'(x)\delta(x-x'),\\
&\{w_2(x), w_2(x')\}_2 = w_2(x)\delta'(x-x') + \partial_x\left(w_2(x)\delta(x-x')\right),
\end{aligned} \tag{81}$$

where $1/\mu$ now plays a role of the deformation parameter. There exists an hierarchy of Hamiltonians related through Lenard relations [3]:

$$\{w_i(x), H_{-j}\}_2 = j\{w_i(x), H_{-j-1}\}_1, \qquad j = 1, 2, 3, \dots. \tag{82}$$

The flows of the bihamiltonian hierarchy are generated by the Hamiltonians H_{-j}, $j < 0$ via:

$$\frac{\partial w_i}{\partial t_{-j+2}} = \{w_i(x), H_{-j}\}_1, \qquad j = 3, \dots, \quad i = 1, 2. \tag{83}$$

The lower Hamiltonians H_{-j} for $j > 1$ can be obtained recursively from the Casimir $H_{-1} = \int w_2(x)\,\mathrm{d}x$ of the first Poisson bracket applying the Lenard relations (82). Following [3, 4], we introduce objects φ_1, φ_2 defined by $w_1 = \varphi_1 - \varphi_{1,x}/\mu$, $w_2 = \varphi_2 + \varphi_{2,x}/\mu$. Then

$$\{\varphi_1(x), w_2(x')\}_1 = \delta'(x-x'), \qquad \{w_1(x), \varphi_2(x')\}_1 = \delta'(x-x')$$

and the Lenard relations yield

$$H_{-2} = \int [\varphi_2(\varphi_1 - \varphi_{1,x}/\mu)]\,\mathrm{d}x, \qquad H_{-3} = \frac{1}{2}\int \left[\varphi_2^2 + \varphi_2\varphi_1(\varphi_1 - \varphi_{1,x}/\mu)\right]\mathrm{d}x.$$

Plugging the above H_{-3} into equation of motion (83) for $j = 3$ we obtain

$$\begin{aligned}
(w_1)_t &= \left(\varphi_2 + \frac{1}{2}\varphi_1^2 - \frac{1}{2\mu}\varphi_1\varphi_{1,x}\right)_x,\\
(w_2)_t &= \left(\varphi_1\varphi_2 + \frac{1}{2\mu}\varphi_1\varphi_{2,x}\right)_x,
\end{aligned} \tag{84}$$

where $t = t_{-1}$. Defining u such that $\varphi_1 = 2u$ and ρ such that $w_2 = -\rho^2/\mu^2 + w_1^2\big/4$ or $\rho^2 = w_1^2\mu^2\big/4 - w_2\mu^2$ we can rewrite the above system of equations after a transformation $t \to -t$ as

$$u_t - u_{xxt}/\mu^2 = \rho\rho_x/\mu^2 - 3uu_x + 2u_x u_{xx}/\mu^2 + uu_{xxx}/\mu^2 \tag{85}$$

$$\rho_t = -(u\rho)_x, \tag{86}$$

which agrees with the two-component Camassa–Holm equation. Multiplying equation (85) by μ^2 and taking $\mu^2 \to 0$ yield

$$-u_{xxt} = \rho\rho_x + 2u_x u_{xx} + uu_{xxx} \tag{87}$$

corresponding to equation (26) with $\kappa = 0$.

In order to take the $\mu \to 0$ limit of the Poisson structure (81) it is convenient to change the variables from w_1, w_2 to m and ρ defined as

$$\begin{aligned}
m &= \tfrac{1}{2}\mu^2(w_1(x) + w_{1,x}/\mu) = \mu^2 u - u_{xx}\\
\rho^2 &= \mu^2\left(w_1^2/4 - w_2\right).
\end{aligned} \tag{88}$$

In terms of m and ρ the Poisson bracket structure (81) turns into

$$\begin{aligned}
&\{m(x), m(x')\}_1 = 0, \\
&\{\rho^2(x), \rho^2(x')\}_1 = -\mu^2(2m(x)\delta'(x-y) + m'(x)\delta(x-y)), \\
&\{\rho^2(x), m(x')\}_1 = \tfrac{1}{2}\mu^2(-\mu^2\delta'(x-x') + \delta'''(x-x')), \\
&\{m(x), m(x')\}_2 = \tfrac{1}{2}\mu^2(\mu^2\delta'(x-x') - \delta'''(x-x')), \\
&\{\rho(x), \rho(x')\}_2 = -\tfrac{1}{2}\mu^2\delta'(x-x'), \\
&\{\rho(x), m(x')\}_2 = 0.
\end{aligned} \tag{89}$$

5.2. The $\mu \to 0$ limit and the Dym-type hierarchy

Redefining the brackets as follows:

$$\{\cdot,\cdot\}_j \longrightarrow \mu^2\{\cdot,\cdot\}_j$$

and taking $\mu \to 0$ limit in equation (89) we find for the first and second bracket structures in terms of u and ρ (see also [15, 19]):

$$\begin{aligned}
&\{u(x), u(x')\}_1 = 0, \\
&\{\rho^2(x), \rho^2(x')\}_1 = 2u''(x')\delta'(x-x') - u'''(x')\delta(x-x') \\
&\{\rho^2(x), u(x')\}_1 = -\tfrac{1}{2}\delta'(x-x'), \\
&\{u(x), u(x')\}_2 = -\tfrac{1}{2}\partial_x^{-1}\delta(x-x'), \\
&\{\rho(x), \rho(x')\}_2 = -\tfrac{1}{2}\partial_x\delta(x-x'), \\
&\{u(x), \rho(x')\}_2 = 0.
\end{aligned} \tag{90}$$

The first bracket in (90) has the Casimir:

$$H_{-1}^{(1)} = \int \left[\rho^2(x) - u_x^2(x)\right] \mathrm{d}x.$$

This Casimir leads via the Lenard relation (82) to the Hamiltonian:

$$H_{-2} = 2\int \left(\rho^2 - u_x^2\right) u \, \mathrm{d}x,$$

which in turn generates equations of motion of $\mu = 0$ case via equations (83) and (82):

$$\begin{aligned}
\frac{\partial u_x}{\partial t} &= \frac{1}{2}\{u_x(x), H_{-2}\}_2 = -\frac{1}{2}(u_x)^2 - uu_{xx} - \frac{1}{2}\rho^2 \\
\frac{\partial \rho}{\partial t} &= \frac{1}{2}\{\rho(x), H_{-2}\}_2 = -(u\rho)_x\,.
\end{aligned} \tag{91}$$

This Hamiltonian structure can be extended by an additional term:

$$\bar{H}_{-2} = -2\kappa \int u^2 \mathrm{d}x.$$

Adding this term to H_{-2} will lead via relations (91) to correct equations of motion (29), (30).

5.3. Hamiltonians of positive order

There exists another class of conserved charges, different from the chain of Hamiltonians H_{-j}, $j = 1, 2, \ldots$ of negative order discussed above. These are the Hamiltonians of positive order originating from the Casimir

$$H_{-1}^{(2)} = 2\int \rho(x)\, \mathrm{d}x \tag{92}$$

of the second Poisson bracket (89). We now employ Lenard relations (82) to construct higher order Hamiltonians. The first recurrence step

$$\left\{\cdot, \int \rho(x)\,\mathrm{d}x\right\}_1 = \{\cdot, H_0\}_2,$$

where '·' stands for phase space variables $m(x)$ and $\rho(x)$, leads to a new Hamiltonian:

$$H_0 = -\int \frac{m(x)}{\rho(x)}\,\mathrm{d}x \tag{93}$$

in agreement with the expression found in [4]. The integrand of H_0 can be rewritten as

$$\frac{m}{\rho} = \rho\frac{m}{\rho^2} = \rho P = \rho f_y = f_x$$

and therefore H_0 appears to be a surface term that would vanish if f would be a local field.

On the next level we find from the Lenard relations

$$\{\cdot, H_0\}_1 = \{\cdot, H_1\}_2,$$

with

$$H_1 = \int \left[\frac{1}{4\rho^3}\left(\rho_x^2 - m^2\right) + \frac{\mu^2}{4\rho}\right]\mathrm{d}x. \tag{94}$$

This recurrence process can be continued to yield higher order Hamiltonians. Technical calculations involved in obtaining higher order Hamiltonians become increasingly tedious. Remarkably, we can bypass these difficulties by relying on the underlying AKNS structure governing higher positive flows. We recall the Hamiltonian densities $\mathcal{H}_n$ (8) of the AKNS model generating the positive flows of the model. Their conservation law with respect to the negative flow s takes a form

$$(\mathcal{H}_n)_s = X_y, \tag{95}$$

where s and y are 'reciprocal' variables describing time and space of the AKNS model and X is some local quantity. The above relation ensures that $\left(\int \mathcal{H}_n\,\mathrm{d}y\right)_s = 0$ (for X local in u and ρ) and thus the integral $\int \mathcal{H}_n\,\mathrm{d}y$ is conserved.

In terms of the original t, x variables the conservation laws (95) read

$$\left(\frac{\partial}{\partial t} + \rho u\frac{\partial}{\partial y}\right)\mathcal{H}_n = \frac{\partial}{\partial y}X$$

or

$$\frac{\partial}{\partial t}\mathcal{H}_n = -\rho u\frac{\partial}{\partial y}\mathcal{H}_n + \frac{\partial}{\partial y}X = -u\frac{\partial}{\partial x}\mathcal{H}_n + \frac{1}{\rho}\frac{\partial}{\partial x}X,$$

where we used that $\partial/\partial y = \rho\partial/\partial x$. It follows that

$$\frac{\partial}{\partial t}(\rho\mathcal{H}_n) = -(u\rho)_x\mathcal{H}_n - u\rho\mathcal{H}_{nx} + X_x = \frac{\partial}{\partial x}(X - u\rho H_n). \tag{96}$$

Thus, the quantities $\rho\mathcal{H}_n$ are conserved charges of the two-component Camassa–Holm and two-component Dym-type models.

The first two Hamiltonian densities rq and rq_y of the AKNS model, given by relations (9) and (10), give rise, after use of definitions (47), (36) and (40), to the following conserved charges:

$$\begin{aligned}
H_1 &= \int \rho\mathcal{H}_1\,\mathrm{d}x = \int \rho r q\,\mathrm{d}x = \int \left[\frac{1}{4\rho^3}\left(\rho_x^2 - m^2\right) + \frac{\mu^2}{4\rho}\right]\mathrm{d}x,\\
H_2 &= \frac{1}{2}\int \rho\mathcal{H}_2\,\mathrm{d}x = \frac{1}{2}\int \rho r q_y\,\mathrm{d}x = \frac{1}{2}\int r q_x\,\mathrm{d}x\\
&= \int \frac{m}{2\rho^2}\left[\frac{\mu^2}{4\rho} - \frac{3}{4}\frac{\rho_x^2}{\rho^3} + \frac{\rho_{xx}}{2\rho^2} - \frac{m^2}{4\rho^3}\right]\mathrm{d}x,
\end{aligned} \tag{97}$$

which we have verified explicitly to be conserved under equations of motion (19), (20) for $m = \mu^2 u - u_{xx}$ for $\mu \neq 0$ and $m = -u_{xx} + \kappa/2$ for $\mu = 0$.

We recognize in $\rho\mathcal{H}_1$ the Hamiltonian H_1 derived in (94) from the Casimir of the second bracket via Lenard recursion relations. Furthermore, we have shown that $H_1 = \int \rho\mathcal{H}_1 \, \mathrm{d}x$ and $H_2 = \int \rho\mathcal{H}_2 \, \mathrm{d}x/2$ also are interrelated via the Lenard relation:

$$\{\cdot, H_1\}_1 = \{\cdot, H_2\}_2.$$

Therefore, the conclusion is that the AKNS-induced Hamiltonians $\rho\mathcal{H}_n$ form the sequence of positive-order Hamiltonians of the 2-component Camassa–Holm- and two-component Dym-type hierarchies. Formula (8) given in section 2 can be used to systematically derive all the Hamiltonians governing positive flows of this model.

Acknowledgments

We thank A Das and L A Ferreira for helpful discussions. HA acknowledges support from Fapesp and thanks the IFT-UNESP Institute for its hospitality. JFG and AHZ thank CNPq for a partial support.

References

[1] Camassa R and Holm D D 1993 *Phys. Rev. Lett.* **71** 1661
[2] Chen M, Liu S-Q and Zhang Y 2005 A 2-component generalization of the Camassa–Holm equation and its solutions *Preprint* nlin.SI/0501028
[3] Liu S Q and Zhang Y 2005 *J. Geom. Phys.* **54** 427 (*Preprint* math-dg/0405146)
[4] Falqui G 2005 On a Camassa–Holm type equation with two dependent variables *J. Phys. A: Math. Gen.* **39** 327–42 (*Preprint* nlin.SI/0505059)
[5] Kruskal M D 1975 Nonlinear wave equations *Dynamical Systems, Theory and Applications (Lecture Notes in Physics* vol 38*)* ed J Moser (New York: Springer)
[6] Cewen C 1990 *Acta Math. Sin.* **6** 35
[7] Hunter J and Zheng Y 1994 *Physica* D **79** 361
[8] Alber M S, Camassa R, Fedorov Yu, Holm D D and Marsden J E 1999 *Phys. Lett.* A **264** 171
[9] Alber M S, Camassa R, Holm D D and Marsden J E 1995 PDEs *Proc. R. Soc.* **450** 677
[10] Brunelli J C, Das A and Popowicz Z 2004 *J. Math. Phys.* **45** 2646 (*Preprint* nlin.si/0307043)
[11] Aratyn H, Gomes J F, Nissimov E, Pacheva S and Zimerman A H 2001 Symmetry flows, conservation laws and dressing approach to integrable models *Integrable Hierarchies and Modern Physical Theories* ed H Aratyn and A Sorin (Dordrecht: Kluwer) pp 243–75
[12] Aratyn H, Ferreira L A, Gomes J F and Zimerman A H 2000 *J. Phys. A: Math. Gen.* **33** L331 (*Preprint* nlin.si/0007002)
[13] Aratyn H, Gomes J F and Zimerman A H 2004 *Nucl. Phys.* B **676** 537 (*Preprint* hep-th/0309099)
[14] Aratyn H, Gomes J F and Zimerman A H 2003 *J. Geom. Phys.* **46** 21
Aratyn H, Gomes J F and Zimerman A H 2003 *J. Geom. Phys.* **46** 201 (*Preprint* hep-th/0107056) (erratum)
[15] Pavlov M V 2005 *J. Phys. A: Math. Gen.* **38** 3823 (*Preprint* nlin.SI/0412072)
[16] Rosenau P 1994 *Phys. Rev. Lett.* **73** 1737
[17] Aratyn H, Ferreira L A, Gomes J F and Zimerman A H 1998 *J. Phys. A: Math. Gen.* **31** 9483 (*Preprint* solv-int/9709004)
Cabrera-Carnero I, Gomes J F, Gueuvoghlanian E P, Sotkov G M and Zimerman A H 2001 *Contribution to 7th Int. Wigner Symp. (Wigsym 7) (College Park, MD, 24–29 August 2001)* (*Preprint* hep-th/0109117)
[18] Camassa R and Zenchuk A I 2001 *Phys. Lett.* A **281** 26
Kraenkel R A and Zenchuk A I 1999 *J. Phys. A: Math. Gen.* **32** 4733
[19] Dubrovin B A and Novikov S P 1993 Hydrodynamics of soliton lattices *Soviet Scientific Reviews, Section C: Mathematical Physics Reviews* vol 9 part 4 (Yverdon: Harwood Academic) 136 pp (as mentioned in [15])